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Stability of Global Solution for the Relativistic Enskog Equation near Vacuum

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Abstract The Cauchy problem of the relativistic Enskog equation with near-vacuum data is considered in this paper. Under the same assumption as that in Jiang (J. Stat. Phys. 127:805–812, 2007) for the relativistic Enskog equation, we obtain the uniform L^∞-stability of the solution. What's more important, is that for two new types of the scattering cross section σ, we give the global existence and $L^1(x,v)$-stability for mild solution when the initial data lies in the space $L^1(x,v)$. As a corollary, we have a BV-type estimate. It is worth mentioning that the stability results in this paper can be applied to the case in Jiang (J. Stat. Phys. 127:805–812, 2007).

Keywords Relativistic Enskog equation · Cauchy problem · (Weighted) stability · Vacuum

1 Introduction

We consider the Cauchy problem for the relativistic Enskog equation

$$\begin{cases} \frac{\partial f}{\partial t} + \hat{v} \cdot \nabla_x f = Q(f,f) \ (x, v \in \mathbb{R}^3, t > 0), \\ f(0,x,v) = f_0(x,v), \end{cases} \tag{RE}$$

where $\hat{v}$, the relativistic velocity, is defined in terms of the momenta $v \in \mathbb{R}^3$ by

$$\hat{v} = \frac{v}{v_0}; \quad v_0 =: \sqrt{1+|v|^2} \tag{1.1}$$

and thus $\hat{v} < 1$ for all v. $f = f(t,x,v)$ denotes the number density of particles with a space position x and a velocity v at time t, and $Q(f,f)$ is the Enskog collision operator.

The Boltzmann equation is one of the most important equations in statistical mechanics, which gives an approximate description of the real behavior of a system of classical particles with short range interaction. While, the Boltzmann equation is no longer valid for

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moderately or highly dense gases. As a modification of the Boltzmann equation, the Enskog equation proposed by Enskog [13] in 1921 is usually used to explain the dynamical behavior of the density profile of moderately or highly dense gases. In fact, different to the Boltzmann equation, in a binary collision of two rigid spheres, the "Enskog" collision does not take place at the same point for the two spheres; Enskog assumed that the two distribution functions at points separated by the distance a equal to the diameter of a sphere. For the background of Boltzmann equation and Enskog equation, one can refer to [4, 5, 13] and references therein.

There are many important works on the classical (non-relativistic) Boltzmann equation. For the mathematical theories on this equation and general references, readers can refer to [5, 7, 10, 15, 22, 27, 33, 34], and the references therein also for the physics backgrounds.

To concentrate on the problems considered in this paper, in the following, we mainly mention some works on the Cauchy problem for the Enskog equation in infinite vacuum. Local existence of solutions to the Enskog equation was studied in [26], while global existence was given in [8, 28, 32]. And Polewczak [28] obtained the existence of a classical solution. The L^1-stability of mild and classical solution for the Boltzmann-Enskog equation ($Y^{\pm} = 1$) was obtained by Cercignani [8] and Ha [21], respectively. For other works about the modified Enskog equation such as large initial data, normalized solutions, convergence of the solutions of Enskog equation to those of the Boltzmann equation, one can refer to [1, 2, 6, 14, 29, 30] and references therein. For further references, one can see a book [3].

For background on the relativistic equation we mention the books of Cercignani et al. [6], deGroot et al. [9] and Stewart [31]. The near-equilibrium relativistic Boltzmann equation admits global smooth solutions [17, 18], and weak solutions are considered in [12].

Although the classical (non-relativistic) Boltzmann equation near vacuum has been heavily studied, the relativistic case has received scant attention. Recently, Glassey [16] proved global existence of solution to the relativistic Boltzmann equation near vacuum. As the author stated in [16], the classical near-vacuum problem was solved in the hard-sphere case via a beautiful trick used in [23, 25] (essentially Galilean invariance: $|x - tv'|^2 + |x - tu'|^2 = |x - tv|^2 + |x - tu|^2$). It allows one to eliminate in many situations the dependence in estimates on the post-collisional velocities u', v'. But that device does not work in the relativistic case. Thus the author in [16] introduced a new weight function for the relativistic Boltzmann equation, which avoids estimating the terms $\hat{v} - \hat{v'}$ and $\hat{v} - \hat{u'}$ in the collision term. This is the key observation for the estimates of the collision term. More recently, Jiang [24] extended the result in [16] to the relativistic Enskog equation under the same assumption on the scattering cross section $\sigma(u, v, \omega)$. However, this new weight function brings us new difficulty to obtain a $L^1(x, v)$-stability of the solution for the relativistic case.

In the present paper, we give the global existence for (RE) using the method in [8] when the initial data is small and belongs to $L^1(x, v)$. In addition, the $L^1(x, v)$-stability and BV-type estimate of the solution in this paper are obtained. On the other hand, using a weighted method together with the method in [8], we have another similar existence and L^1-stability under a more general assumption (H2) on σ than that in [24]. Lastly, we prove the solution established in [24] is L^∞-stability under the same condition on the scattering cross section σ as in [24] when the initial data has not compact support in x, i.e., the solution in [24] does not belong to $L^1(x, v)$.

To our best knowledge, this is the first work on the stability of solutions for the relativistic Enskog equation near vacuum. Furthermore, The existence result in Jiang [24] is extended to those with some new types of the scattering cross section σ, at the same time, we obtain the uniform L^1-stability and L^∞-stability.

Remark 1.1 The method using in Sect. 3 can't be used to the relativistic Boltzmann equation, although the initial data has compact support with respect to x, i.e., the solution lies in $L^1(x,v)$. We can't obtain a similar estimate as Lemma 3.2 for the relativistic Boltzmann equation since the function space of the solution in [16] is indeed a L^∞ type.

Remark 1.2 L^1-type of estimates for the Boltzmann equation are also important in the context of the Boltzmann theory since L^1 seems to be the natural space where weak solutions lie in. Near vacuum solutions naturally lie in the space bounded by Maxwellian distribution. By constructing some Lyapunov functionals measuring the future collision, Ha in [19, 20] builded the L^1-type of estimates of classical solutions for the Boltzmann equation. It mainly depends on the decay rate of the collision terms and Gronwall's inequality. We can't obtain the uniform $L^1(x,v)$-stability in time of the solution for the relativistic Boltzmann equation in [16] using the method in [11, 20], since we can't get the temporal decay of the collision terms using the weight function as $e^{-\sqrt{1+|v|^2}}e^{-|x\times v|}$ for the relativistic Boltzmann equation, which is different to $e^{-|x|^2-|v|^2}$ for the non-relativistic one. Thus we only obtain a non-uniform $L^1(x,v)$-stability result as $\|f(t,x,v)-\bar{f}(t,x,v)\|_{L^1(x,v)}\le Ce^t\|f_0(x,v)-\bar{f}_0(x,v)\|_{L^1(x,v)}$ for the solutions in [16]. The details can be found in the end of this paper.

The rest of this paper will be organized as follows. First of all, in Sect. 2, we explicitly write the equation and some assumptions and main results in this paper. In Sect. 3, we prove the global existence and $L^1(x,v)$-stability of the mild solution. Lastly, in Sect. 4, the L^∞-stability of the mild solution in [24] is obtained.

2 Formulations and Main Result

As is standard, we use primes to represent the post-collision velocities. The conservation laws for momentum and energy are

$$u'+v'=u+v\equiv\mathfrak{m}, \tag{2.1}$$

$$\sqrt{1+|u'|^2}+\sqrt{1+|v'|^2}=\sqrt{1+|u|^2}+\sqrt{1+|v|^2}\equiv\mathfrak{e}. \tag{2.2}$$

Let us begin by defining the remaining variables in the collision integral. We define

$$\begin{aligned} s&=(U+V)^2=(u_0+v_0)^2-|u+v|^2=2u_0v_0-2u\cdot v+u_0^2+v_0^2-|v|^2\\ &=2(\sqrt{1+|u|^2}\sqrt{1+|v|^2}-u\cdot v+1), \end{aligned} \tag{2.3}$$

$$\begin{aligned} 4g^2&=-(U-V)^2=-(u_0-v_0)^2+|u-v|^2=2u_0v_0-2u\cdot v-u_0^2+|u|^2-v_0^2+|v|^2\\ &=2(\sqrt{1+|u|^2}\sqrt{1+|v|^2}-u\cdot v-1), \end{aligned} \tag{2.4}$$

and

$$\cos\theta=\frac{(V-U)\cdot(V'-U')}{(V-U)^2}. \tag{2.5}$$

Furthermore, we define the Møller velocity as the scalar v_M given by

$$v_M^2=|\hat{v}-\hat{u}|^2-|\hat{v}\times\hat{u}|^2=\frac{s(s-4)}{4v_0^2u_0^2}$$

or

$$v_M = \frac{2g\sqrt{1+|g|^2}}{v_0 u_0}. \tag{2.6}$$

There are several representations of the collision operator Q; we will use that in Glassey [16], which also was used in Appendix II of [17]. So we write the collision operator as follows.

$$\begin{aligned} Q(f,f) &= a^2 \int_{\mathrm{S}^2_+} \int_{\mathbb{R}^3} q(u,v,\omega)[Y(\rho_+) f(t,x,v') f_+(t,x,u') \\ &\quad - Y(\rho_-) f(t,x,v) f_-(t,x,u)] du d\omega \\ &= Q_+(f,f) - Q_-(f,f), \end{aligned} \tag{2.7}$$

where $\mathrm{S}^2_+ = \{\omega \in \mathrm{S}^2 : \ \omega \cdot \hat{v} \geq \omega \cdot \hat{u}\}$. And

$$\begin{aligned} &\rho(t,x) = \int_{\mathbb{R}^3} f(t,x,v) dv, \\ &\rho_+ = \rho(t, x + a\omega), \qquad \rho_- = \rho(t, x - a\omega), \\ &f(u) = f(t,x,u), \qquad f_+(u) = f(t, x + a\omega, u), \qquad f_-(u) = f(t, x - a\omega, u). \end{aligned}$$

If the factors $Y^{\pm} = Y(\rho_{\pm})$ are set to be the same positive constant and the diameter a equal to zero in the density variables, then the relativistic Enskog equation becomes the relativistic Blotzmann one.

Furthermore, as [16, 17], the scattering kernel q is

$$q(u,v,\omega) = \frac{4s\mathfrak{e}^2 |(\omega \cdot (\hat{v} - \hat{u})|}{(\mathfrak{e}^2 - (\omega \cdot \mathfrak{m})^2)^2} \sigma, \tag{2.8}$$

where σ is the scattering cross section. Then we state the assumptions on σ:

(H0) [16, 24] *Hypothesis on the scattering cross section* σ Let ω be a unit vector, $u, v \in \mathbb{R}^3$ and $0 < \delta < 1$. $\sigma = \sigma(u,v,\omega)$ is to be nonnegative, continuous and satisfy

$$0 \leq \sigma(u,v,\omega) \leq \frac{|\omega \cdot (\hat{v} \times u)| \tilde{\sigma}(\omega)}{g(1+|g|^2)^{\frac{1}{2}+\delta}}, \tag{2.9}$$

where $\tilde{\sigma}(\omega)$ is also nonnegative, bounded and continuous and satisfies for some positive constant b and for every $0 \neq z \in \mathbb{R}^3$,

$$\int_{|\omega|=1} \frac{\tilde{\sigma}(\omega)}{1 + |\omega \cdot z|} d\omega \leq b|z|^{-1}. \tag{2.10}$$

Discussion of (H0) [16, 24] An assumption in (H0) on the decay in g is natural because of the following inequality [16]:

$$|q(u,v,\omega)| \leq 8g\sqrt{1+|g|^2}\sigma, \tag{2.11}$$

the factor $\omega \cdot (\hat{v} \times u)$ appearing in (H0) plays a vital role in the estimates on the gain term in [16, 24]. As the author remarked in [16], we do not know if the factor represents a physically realistic situation.

Assume that $Y^{\pm}$ is locally Lipschitz, i.e.,

$$|Y^{\pm}(p) - Y^{\pm}(r)| \leq C(R)\|p - r\|_M, \quad \text{for any } p, r \in M_R, \tag{2.12}$$

where $\|\cdot\|_M$ is the norm of function space of the solution, and $M_R = \{f : \|f\|_M \leq R, R > 0\}$. And $C(R)$ is a positive nondecreasing function on $\mathbb{R}_+$.

Next, we state the global existence of the mild solution for (RE) in [24].

Theorem 1 *Let σ, δ satisfy the Hypothesis* (H0) *and let $Y^{\pm}$ be as in* (2.12). *Consider* (RE) *with initial value $f_0(x, v)$ satisfying $0 \leq f_0(x, v) \in C^0(\mathbb{R}^6)$ as well as*

$$e^{v_0}[1 + |x \times v|^2]^{\frac{1+\delta}{2}} f_0(x, v) \leq c_0.$$

Then there exists a positive number ϵ with the property that if $c_0 \leq \epsilon$, a uniquely determined nonnegative global solution $f(t, x, v)$ to the mild form of the Cauchy problem for (RE) *exists. This solution satisfies the estimate*

$$f(t, x, v) \leq ce^{-v_0}[1 + |x \times v|^2]^{-\frac{1+\delta}{2}},$$

where c is a constant depending only on the initial data.

Now, we give the main results in this paper.

(H1) *Hypothesis on the scattering cross section σ* Let ω be a unit vector, $u, v \in \mathbb{R}^3$. $\sigma = \sigma(u, v, \omega)$ is to be nonnegative, continuous and satisfy

$$\sigma(u, v, \omega) =: \sigma(\mathfrak{e}, \mathfrak{m}, \omega) \leq k\frac{(\mathfrak{e}^2 - (\omega \cdot \mathfrak{m})^2)^2}{4(\mathfrak{e}^2 - |\mathfrak{m}|^2)\mathfrak{e}^2}, \quad \text{for some } k > 0,$$

which implies

$$|q(u, v, \omega)| \leq k|\omega \cdot (\hat{v} - \hat{u})|. \tag{2.13}$$

In fact, (2.13) is a key point in the proof of Theorems 2 and 3.

Especially, from the relation in [16]

$$|q(u, v, \omega)| \leq 4u_0v_0|\omega \cdot (\hat{v} - \hat{u})|\sigma(u, v, \omega),$$

equation (2.13) is true when $|\sigma(u, v, \omega)| \leq \frac{1}{u_0v_0}$.

We introduce the following function space:

$$M = \left\{ f(t, x, v) : f \text{ defined in } [0, T] \times \mathbb{R}^6 \right.$$
$$\left. \text{with } \frac{\partial f}{\partial t} + \hat{v} \cdot \nabla_x f \in L^1([0, T] \times \mathbb{R}^6);\ f_0(x, v) \in L^1(\mathbb{R}^6) \right\}$$

with norm

$$\|f\|_M = \|f_0\|_{L^1(\mathbb{R}^6)} + \left\| \frac{\partial f}{\partial t} + \hat{v} \cdot \nabla_x f \right\|_{L^1([0,T]\times\mathbb{R}^6)}.$$

Obviously, M is a Banach space.

Theorem 2 *Let σ satisfy the Hypothesis* (H1) *and $Y^{\pm}$ satisfy* (2.12). *Consider* (RE) *with initial value $f_0(x,v) \in L^1(\mathbb{R}^6)$ as well as*

$$\|f_0(x,v)\|_{L^1(\mathbb{R}^6)} \le R_0, \ R_0 \text{ is sufficiently small.}$$

Then there exists a unique global solution $f(t,x,v)$ to the mild form of the Cauchy problem for (RE). *This solution satisfies the estimate*

$$\left\| \frac{\partial f}{\partial t} + \hat{v} \cdot \nabla_x f \right\|_{L^1([0,T]\times\mathbb{R}^6)} \le C \|f_0(x,v)\|_{L^1(\mathbb{R}^6)},$$

where C is a positive constant independent of the variables t, x, v.

Theorem 3 *Let σ satisfy the Hypothesis* (H1) *and $Y^{\pm}$ satisfy* (2.12). *Consider two mild solutions $f, \bar{f} \in M$ with the initial data $f_0, \bar{f}_0$, respectively. Here M is defined as that in Theorem* 2. *Then we have the $L^1(x,v)$-stability of the two mild solutions*:

$$\|(f-\bar{f})(t,x,v)\|_{L^1(x,v)} \le C \|f_0(x,v) - \bar{f}_0(x,v)\|_{L^1(x,v)},$$

where C is a uniform constant with respect to time t.

Next, we give the global existence and $L^1(x,v)$-stability in a weighted space. Choose a weight function $\phi(v) = (\sqrt{1+|v|^2})^{2\alpha+1} = v_0^{2\alpha+1}$, and denote

$$\|f\|_{L^1_\phi(x,v)} = \int_{\mathbb{R}^6} \phi(v) f dx dv.$$

We introduce a weighted function space:

$$M^\phi = \Big\{ f(t,x,v) : \ f \text{ defined in } [0,T]\times\mathbb{R}^6 \\ \text{with } \frac{\partial f}{\partial t} + \hat{v}\cdot\nabla_x f \in L^1_\phi([0,T]\times\mathbb{R}^6); \ f_0(x,v) \in L^1_\phi(\mathbb{R}^6) \Big\}$$

with norm

$$\|f\|_{M^\phi} = \|f_0\|_{L^1_\phi(\mathbb{R}^6)} + \left\| \frac{\partial f}{\partial t} + \hat{v}\cdot\nabla_x f \right\|_{L^1_\phi([0,T]\times\mathbb{R}^6)}.$$

Obviously, M^ϕ is a Banach space. We denote $M^\phi_R = \{f : \ \|f\|_{M^\phi} \le R\}$.

(H2) *Hypothesis on the scattering cross section σ*

$$\sigma(u,v,\omega) \le r \frac{(u_0)^\alpha (v_0)^{\alpha-1}}{(1+|g|^2)^\alpha}, \quad \alpha \ge 0, \quad \text{for some } r > 0.$$

Theorem 4 *Let σ satisfy the Hypothesis* (H2) *and $Y^{\pm}$ satisfy* (2.12). *The initial data $f_0 \in L^1_\phi(x,v)$ is appropriately small. Then there exists a global mild solution $f \in M^\phi$.*

On the other hand, consider two mild solutions $f, \bar{f} \in M^\phi$ with the initial data $f_0, \bar{f}_0 \in L^1_\phi(x,v)$, respectively. Then there exists a global mild solution in M^ϕ for the problem (RE) *satisfies the weighted $L^1(x, v)$-stability of the two mild solutions*:

$$\|(f - \bar{f})(t, x, v)\|_{L^1_\phi(x,v)} \le C\|f_0(x, v) - \bar{f}_0(x, v)\|_{L^1_\phi(x,v)},$$

where C is a uniform constant with respect to time t.

Remark 2.1 Recall the assumption (H0), we know

$$\begin{aligned}\sigma(u, v, \omega) &\le \frac{|\omega\cdot(\hat{v}\times u)|\tilde{\sigma}(\omega)}{g(1+|g|^2)^{\frac{1}{2}+\delta}} \le \frac{2u_0^{\frac{1}{2}}v_0^{\frac{1}{2}}}{[|u\times v|^2+|u-v|^2]^{\frac{1}{2}}}\frac{v_0^{-1}|\omega\cdot(v\times u)|}{(1+|g|^2)^{\frac{1}{2}+\delta}}\\ &\le r_1\frac{u_0^{\frac{1}{2}}v_0^{-\frac{1}{2}}}{(1+|g|^2)^{\frac{1}{2}+\delta}}, \quad \text{for some } r_1 > 0,\end{aligned} \tag{2.14}$$

where we have used the inequality in Lemma 3.1 of [17]:

$$g \ge \frac{[|u\times v|^2+|u-v|^2]^{\frac{1}{2}}}{2u_0^{\frac{1}{2}}v_0^{\frac{1}{2}}}. \tag{2.15}$$

From (2.14), we know the Hypothesis (H2) is weaker than (H0). In other words, we indeed extend the existence result in [24] when the initial data has compact support with respect to x. Moreover, we obtain the $L^1(x, v)$-stability.

In the end, under the same assumption on $\sigma(u, v, \omega)$ as that in [24], and the initial data has not compact support with respect to x, we obtain the following L^∞-stability of solutions.

Theorem 5 *Let σ satisfy the Hypothesis* (H0). *Consider two mild solutions in Theorem 1 $f, \bar{f} \in \tilde{M}$ with the initial data $f_0, \bar{f}_0$, respectively. Then the $L^\infty(x, v)$-stability of the two mild solutions is established*:

$$\|f(t, x, v) - \bar{f}(t, x, v)\| \le C\|f_0(x, v) - \bar{f}_0(x, v)\|_0,$$

where C is a uniform constant with respect to time t, and the norms $\|\cdot\|$, $\|\cdot\|_0$, the space $\tilde{M}$ are defined in Sect. 4.

3 Existence and $L^1(x, v)$-stability

In this section, under the assumption (H1) or (H2) on σ, we prove the global existence and $L^1(x, v)$-stability of the mild solution for (RE).

3.1 Under the Hypothesis (H1)

Lemma 3.1 *If $f \in M$, then there is a function $l \in L^1(\mathbb{R}^3\times\mathbb{R}^3)$*

$$|f(t, x, v)| \le l(x - \hat{v}t, v), \tag{3.1}$$

$$\|l\|_{L^1(\mathbb{R}^3\times\mathbb{R}^3)} \le \|f\|_M. \tag{3.2}$$

Proof Let

$$P(t,x,v)=\frac{\partial f}{\partial t}+\hat{v}\cdot\nabla_x f,$$

one finds

$$f(t,x,v)=f_0(x-\hat{v}t,v)+\int_0^t P(s,x-\hat{v}t+\hat{v}s,v)ds. \tag{3.3}$$

Define

$$l(x,v)=|f_0(x,v)|+\int_0^T |P(s,x+\hat{v}s,v)|ds, \tag{3.4}$$

which implies (3.1). Then from the definition of the norm in M in Sect. 2, we get (3.2). This proves Lemma 3.1. □

Lemma 3.2 *If $f\in M_R$, then $Q(f,f)\in L^1([0,T]\times\mathbb{R}^6)$ and*

$$\|Q(f,f)\|_{L^1([0,T]\times\mathbb{R}^6)}\le C_2(R)\|f\|_M^2, \tag{3.5}$$

where R is given in the definition of the space M_R and $C_2(R)$ is a positive nondecreasing function on $\mathbb{R}_+$.

Proof Firstly, we consider the loss term $Q_-(f,f)$. From the assumption (2.12) on $Y^\pm$, we have

$$|Y^\pm(f)|\le L(R)R+|Y^+(0)|+|Y^-(0)|:=C_1(R), \tag{3.6}$$

where $C_1(R)$ is a positive nondecreasing function on $\mathbb{R}_+$. Then

$$\begin{aligned}
&\|Q_-(f,f)\|_{L^1([0,T]\times\mathbb{R}^6)}\\
&\quad=a^2\int_{\mathbb{S}^2_+\times\mathbb{R}^9\times\mathbb{R}_+}Y^-(\rho)q(u,v,\omega)f(t,x,v)f(t,x-a\omega,u)dud\omega dvdxdt\\
&\quad\le C_1(R)a^2\int_{\mathbb{S}^2_+\times\mathbb{R}^9\times\mathbb{R}_+}q(u,v,\omega)l(x-\hat{v}t,v)l(x-a\omega-\hat{u}t,u)dud\omega dvdxdt\\
&\quad\le C_1(R)ka^2\int_{\mathbb{S}^2_+\times\mathbb{R}^9\times\mathbb{R}_+}|\omega\cdot(\hat{v}-\hat{u})|l(x-\hat{v}t,v)l(x-a\omega-\hat{u}t,u)dud\omega dvdxdt\\
&\quad\le C_1(R)ka^2\int_{\mathbb{S}^2_+\times\mathbb{R}^9\times\mathbb{R}_+}l(y,v)l(y+(\hat{v}-\hat{u})t-a\omega,u)|\omega\cdot(\hat{v}-\hat{u})|dud\omega dvdxdt\\
&\quad=C_1(R)k\int_{\mathbb{R}^6\times\mathbb{R}^6}l(y,v)l(z,u)dydvdzdu\\
&\quad\le C_2(R)\|l\|^2_{L^1(\mathbb{R}^6)}.
\end{aligned} \tag{3.7}$$

Here

$$y=x-\hat{v}t,\qquad z=y+(\hat{v}-\hat{u})t-a\omega,$$

and we have used the assumption on $\sigma(u,v,\omega)$: (H1), and the fact

$$\frac{\partial z}{\partial(t,\omega)}=a^2|\omega\cdot(\hat{v}-\hat{u})|. \tag{3.8}$$

On the other hand, the property in [17]

$$\frac{\partial(u', v')}{\partial(u, v)} = -\frac{\sqrt{1+|u'|^2}\sqrt{1+|v'|^2}}{\sqrt{1+|u|^2}\sqrt{1+|v|^2}}, \tag{3.9}$$

which together with (2.8) yields

$$q(u, v, \omega) = q(u', v', \omega)\frac{\partial(u', v')}{\partial(u, v)}. \tag{3.10}$$

Then using the change of variables $(v', u') \to (v, u)$ and the same method as above, we can obtain the same estimate for the gain term $Q_+(f, f)$ as that of Q_-. In the end, from (3.7) and Lemma 3.1, we complete the proof of Lemma 3.2. □

Up to now, it is easy to prove a small-data existence result with the quadratic estimates of the form from Lemma 3.2 in hand.

Define the operator $\mathcal{F}:\ h \to f$ on M_R as follows:

Given $f_0(x, v) \in L^1(x, v)$ and $h \in M_R$, f is a solution of

$$\frac{\partial f}{\partial t} + \hat{v}\cdot\nabla_x f = Q(h, h), \quad f(0, x, v) = f_0(x, v). \tag{3.11}$$

By Lemma 3.2, we have

$$\|Q(h, h)\|_{L^1([0,T]\times\mathbb{R}^6)} \le C(R)\|h\|_M^2. \tag{3.12}$$

Hence

$$\left\|\frac{\partial f}{\partial t} + \hat{v}\cdot\nabla_x f\right\|_{L^1([0,T]\times\mathbb{R}^6)} \le C(R)\|h\|_M^2, \tag{3.13}$$

and

$$\|f\|_M \le \|f_0\|_{L^1(\mathbb{R}^6)} + C(R)\|h\|_M^2. \tag{3.14}$$

So $\mathcal{F}$ maps the ball B_R defined by $\|h\|_M \le R$ into another ball $B_{\bar{R}}$ with

$$\bar{R} \le R_0 + C(R)R^2 \le R, \tag{3.15}$$

if R_0, R are sufficiently small. Then from (3.11) one gets

$$\|f - \bar{f}\|_M \le \|h + \bar{h}\|_M\|h - \bar{h}\|_M \le 2R\|h - \bar{h}\|_M. \tag{3.16}$$

Thus, we find that mapping $\mathcal{F}$ is a strict contraction if R is sufficiently small. Using the standard contraction mapping principle, we get the follow existence result.

Theorem 3.1 *Let σ satisfy the Hypothesis* (H1) *and $Y^{\pm}$ satisfy* (2.12). *Consider* (RE) *with initial value $f_0(x, v) \in L^1(\mathbb{R}^6)$ as well as*

$$\|f_0(x, v)\|_{L^1(\mathbb{R}^6)} \le R_0, \quad R_0 \text{ is sufficiently small}. \tag{3.17}$$

Then there exists a unique global solution $f(t, x, v)$ to the mild form of the Cauchy problem for (RE). *This solution satisfies the estimate*

$$\left\|\frac{\partial f}{\partial t} + \hat{v}\cdot\nabla_x f\right\|_{L^1([0,T]\times\mathbb{R}^6)} \le C\|f_0(x, v)\|_{L^1(\mathbb{R}^6)},$$

where C is a positive constant independent of the variables t, x, v.

Next, we consider the stability of solution in Theorem 3.1.
From the definition of the norm $\|\cdot\|_M$ and (3.3), we find

$$\sup_{0\le t\le T} \|f\|_{L^1(\mathbb{R}^6)} \le \|f\|_M. \tag{3.18}$$

Theorem 3.2 *Let σ satisfy the Hypothesis* (H1) *and $Y^{\pm}$ satisfy* (2.12). *If two initial data $f_0, \bar{f}_0 \in L^1(\mathbb{R}^6)$ satisfy* (3.17), *then the corresponding solutions are $L^1(x,v)$-stability*:

$$\sup_{t\in\mathbb{R}^+} \|f-\bar{f}\|_{L^1(\mathbb{R}^6)} \le C\|f_0-\bar{f}_0\|_{L^1(\mathbb{R}^6)}, \tag{3.19}$$

where C is a positive constant independent of the variables t, x, v.

Proof As (3.16), we have

$$\|f-\bar{f}\|_M \le \|f_0-\bar{f}_0\|_{L^1(\mathbb{R}^6)} + C(R)\|f+\bar{f}\|_M\|f-\bar{f}\|_M. \tag{3.20}$$

And

$$C(R)\|f+\bar{f}\|_M \le 2C(R)R < 1,$$

if let R be small enough. Thus we get

$$\|f-\bar{f}\|_M \le \frac{1}{1-2C(R)R}\|f_0-\bar{f}_0\|_{L^1(\mathbb{R}^6)}, \tag{3.21}$$

which together with (3.18) implies (3.19). This proves Theorem 3.2. □

As a corollary of the uniform L^1 stability for the mild solutions we can obtain a uniform BV-type estimates. We define difference quotients as follows.

$$D_{x_i}^h f(t,x,v) = \frac{f(t,x+he_i,v)-f(t,x,v)}{h},$$

where $h>0$ is any fixed constant and e_i is an orthonormal basis of $\mathbb{R}^3$.
And set

$$\|\nabla_x f(t)\|_{L^1(x,v)} := \sum_{i=1}^{3} \|\partial_{x_i} f(t)\|_{L^1(x,v)}.$$

Corollary 3.1 *Suppose the main assumption in Theorem* 3 *holds, and let f be a continuous mild solution corresponding to small initial data $\partial_{x_i} f(0) \in L^1(\mathbb{R}^6)$, $i=1,2,3$. Then there exists a positive constant independent of time t such that*

$$\|D_{x_i}^h f(t,x,v)\|_{L^1(x,v)} \le C\|\nabla_x f(0)\|_{L^1(x,v)}.$$

Furthermore, if $\nabla_x f(t)$ exists, we have BV-type estimate:

$$\|\nabla_x f(t)\|_{L^1(x,v)} \le C\|\nabla_x f(0)\|_{L^1(x,v)}.$$

Here C is a positive constant independent of t, x, v.

Proof From the definition of difference quotients, we find

$$\begin{aligned}\|D^h_{x_i} f(t,x,v)\|_{L^1(x,v)} &= \left\| \frac{f(t,x+he_i,v)-f(t,x,v)}{h}) \right\|_{L^1(x,v)} \quad \text{(by Theorem 3.2)}\\ &\le C\left\| \frac{f(0,x+he_i,v)-f(0,x,v)}{h} \right\|_{L^1(x,v)} \le C\|\partial_{x_i} f(0)\|_{L^1(x,v)},\end{aligned}$$

where we have used the relation between difference quotients and derivatives.

If $\nabla_x f(t)$ exists, we get BV-type estimate:

$$\|\nabla_x f(t)\|_{L^1(x,v)} \le C\|\nabla_x f(0)\|_{L^1(x,v)}.$$

Thus, we complete the proof of Corollary 3.1. □

Remark 3.1

1. The proof of the BV-type estimate here is obviously simpler than that in [11, 20].
2. If one gets a classical solution for the problem (RE), using the functionals constructed in [21], the $L^1(x,v)$-stability of the classical solution for (RE) can be obtained.

3.2 Under the Hypothesis (H2)

In this subsection, we obtain the global existence and L^1-stability under the hypothesis (H2) by choosing the weight function $\phi(v)$ defined in Sect. 2.

Recall the weight function $\phi(v) = (\sqrt{1+|v|^2})^{2\alpha+1} = v_0^{2\alpha+1}$ and the weighted function space:

$$M^\phi = \Big\{ f(t,x,v):\ f \text{ defined in } [0,T]\times\mathbb{R}^6 \\ \text{with } \frac{\partial f}{\partial t} + \hat{v}\cdot\nabla_x f \in L^1_\phi([0,T]\times\mathbb{R}^6);\ f_0(x,v)\in L^1_\phi(\mathbb{R}^6)\Big\}$$

with norm

$$\|f\|_{M^\phi} = \|f_0\|_{L^1_\phi(\mathbb{R}^6)} + \left\|\frac{\partial f}{\partial t} + \hat{v}\cdot\nabla_x f\right\|_{L^1_\phi([0,T]\times\mathbb{R}^6)},$$

where

$$\|f\|_{L^1_\phi(x,v)} = \int_{\mathbb{R}^6} \phi(v) f\, dx\, dv.$$

Obviously, M^ϕ is a Banach space. We denote $M^\phi_R = \{f:\ \|f\|_{M^\phi} \le R\}$.

Lemma 3.3 *If $f \in M^\phi$, then there is a function $l \in L^1(\mathbb{R}^3\times\mathbb{R}^3)$*

$$|F(t,x,v)| = |\phi(v) f(t,x,v)| \le \phi(v) l(x-\hat{v}t, v), \tag{3.22}$$

$$\|\phi(v) l\|_{L^1(\mathbb{R}^3\times\mathbb{R}^3)} \le \|f\|_{M^\phi}. \tag{3.23}$$

Proof Let

$$\phi(v)P(t,x,v)=\frac{\partial F}{\partial t}+\hat{v}\cdot\nabla_x F,$$

one finds

$$F(t,x,v)=F_0(x-\hat{v}t,v)+\int_0^t \phi(v)P(s,x-\hat{v}t+\hat{v}s,v)ds. \tag{3.24}$$

Define

$$l(x,v)=|f_0(x,v)|+\int_0^T |P(s,x+\hat{v}s,v)|ds, \tag{3.25}$$

which implies (3.22). Then from the definition of the norm for the space M^ϕ, we get (3.23). This proves Lemma 3.3. □

Lemma 3.4 *If* $\|f\|_{M^\phi}\le R$, *then* $Q(f,f)\in L^1([0,T]\times\mathbb{R}^6)$ *and*

$$\|\phi(v)Q(f,f)\|_{L^1([0,T]\times\mathbb{R}^6)}\le C_3(R)\|f\|^2_{M^\phi}. \tag{3.26}$$

Here $C_3(R)$ *is a positive nondecreasing function on* $\mathbb{R}_+$.

Proof Firstly, we consider the loss term $Q_-(f,f)$. From the assumption (2.12) on $Y^\pm$, we have

$$|Y^\pm(f)|\le L(R)R+|Y^+(0)|+|Y^-(0)|:=C_1(R), \tag{3.27}$$

where $C_1(R)$ is a positive nondecreasing function on $\mathbb{R}_+$.

By direct calculation, we know

$$\frac{\sqrt{1+|v|^2}}{\sqrt{1+|u|^2}(1+|g|^2)}=\frac{2\sqrt{1+|v|^2}}{\sqrt{1+|u|^2}(\sqrt{1+|v|^2}\sqrt{1+|u|^2}-u\cdot v+1)}\le r_2, \tag{3.28}$$

where r_2 is a positive constant independent of u,v.

From (3.28), the assumption (H2) and the fact in [16]

$$q(u,v,\omega)\le 4u_0v_0\sigma|\omega\cdot(\hat{v}-\hat{u})|, \tag{3.29}$$

we get when $\alpha\ge 0$

$$\begin{aligned}
&\|\phi(v)Q_-(f,f)\|_{L^1([0,T]\times\mathbb{R}^6)}\\
&=a^2\int_{S^2_+\times\mathbb{R}^9\times\mathbb{R}_+} Y^-(\rho)\phi(v)q(u,v,\omega)f(t,x,v)f(t,x-a\omega,u)dud\omega dvdxdt\\
&\le C_1(R)a^2\int_{S^2_+\times\mathbb{R}^9\times\mathbb{R}_+}\phi(v)q(u,v,\omega)l(x-\hat{v}t,v)l(x-a\omega-\hat{u}t,u)dud\omega dvdxdt\\
&\le C_1(R)a^2\int_{S^2_+\times\mathbb{R}^9\times\mathbb{R}_+}\frac{u_0^{\alpha+1}v_0^\alpha}{u_0^{2\alpha+1}(1+|g|^2)^\alpha}|\omega\cdot(\hat{v}-\hat{u})|\phi(v)l(x-\hat{v}t,v)\\
&\quad\times\phi(u)l(x-a\omega-\hat{u}t,u)dud\omega dvdxdt
\end{aligned}$$

$$= C_1(R)a^2 \int_{\mathbb{S}^2_+\times\mathbb{R}^9\times\mathbb{R}_+} \frac{v_0^\alpha}{u_0^\alpha(1+|g|^2)^\alpha} |\omega\cdot(\hat{v}-\hat{u})|\phi(v)l(x-\hat{v}t,v)$$
$$\times \phi(u)l(x-a\omega-\hat{u}t,u)dud\omega dvdxdt$$
$$\le C_3(R)a^2 \int_{\mathbb{S}^2_+\times\mathbb{R}^9\times\mathbb{R}_+} \phi(v)l(y,v)\phi(u)l(y+(\hat{v}-\hat{u})t-a\omega,u)$$
$$\times |\omega\cdot(\hat{v}-\hat{u})|dud\omega dvdxdt$$
$$= C_3(R)\int_{\mathbb{R}^6\times\mathbb{R}^6} \phi(v)l(y,v)\phi(u)l(z,u)dydvdzdu$$
$$\le C_3(R)\|\phi l\|^2_{L^1(\mathbb{R}^6)}. \tag{3.30}$$

Here

$$y = x-\hat{v}t, \qquad z = y+(\hat{v}-\hat{u})t-a\omega,$$

and we have used the fact

$$\frac{\partial z}{\partial(t,\omega)} = a^2|\omega\cdot(\hat{v}-\hat{u})|. \tag{3.31}$$

On the other hand, the property in [17]

$$\frac{\partial(u',v')}{\partial(u,v)} = -\frac{\sqrt{1+|u'|^2}\sqrt{1+|v'|^2}}{\sqrt{1+|u|^2}\sqrt{1+|v|^2}}, \tag{3.32}$$

which together with (2.8) yields

$$q(u,v,\omega) = q(u',v',\omega)\frac{\partial(u',v')}{\partial(u,v)}. \tag{3.33}$$

Then using the change of variables $(v',u')\to(v,u)$ and the same method as above, we can obtain the same estimate for the gain term $Q_+(f,f)$ as that of Q_-. In the end, from (3.30) and Lemma 3.3, we complete the proof of Lemma 3.4. □

Then a same process as in Sect. 3.1 suffices to complete the proof of Theorem 4.

4 L^∞ Stability of the Solution in Theorem 1

In this section, we consider L^∞ stability of two mild solutions obtained in [24]. The reason is stated in the following remark.

Remark 4.1 The solution f in Theorem 1 need not be integrable in x. But if one gives an additional assumption that the initial value $f_0(x,v)$ has compact support in $\{x : |x| \le k, k>0\}$, it can be seen from the representation that the gain and loss terms (and hence the solution f as well) each has support in $\{|x|\le k+t\}$. Therefore the solution is integrable in x and v under this additional assumption. While, without this additional assumption, we only give L^∞ stability of two mild solutions obtained in [24].

The function space of the solution in [24] is as follows. Denote

$$\tilde{M} = \left\{ f \in C^0([0,\infty) \times \mathbb{R}^6) : \|f\| \equiv \sup_{t,x,v} e^{|v_0|}[1 + |x \times v|^2]^{\frac{1+\delta}{2}} |f(t,x,v)| < \infty \right\},$$

and $\|f_0\|_0 = \sup_{x,v}[1+|x \times v|^2]^{\frac{1+\delta}{2}} e^{|v_0|}|f_0(x,v)|$. For simplicity, we denote weight function $W(x,v)$ as

$$W(x,v) = e^{-v_0}[1 + |x \times v|^2]^{-\frac{1+\delta}{2}}.$$

From now on, $f^\sharp(t,x,v) = f(t, x + \hat{v}t, v)$.

Lemma 4.1 *Let hypothesis* (H0) *hold. For any* $t \geq 0$ *and* $f^\sharp, \bar{f}^\sharp \in \tilde{M}$ *there is a constant* $C(R)$ *independent of* t, x, v *for which*

$$\int_0^t |Q^\sharp_\pm(f,\bar{f})|(\tau)d\tau \leq C(R)W(x,v)\|f^\sharp\|\|\bar{f}^\sharp\|. \tag{4.1}$$

Theorem 4.1 *Let* $f_0(x,v)$ *and* $\bar{f}_0(x,v)$ *be two nonnegative functions such that* $\max\{\|f_0\|_0, \|\bar{f}_0\|_0\} < \frac{1}{9C(R)}$, *and suppose that* f *and* $\bar{f}$ *are the nonnegative solutions for the Cauchy problem* (RE) *to initial value* f_0 *and* $\bar{f}_0$, *respectively. Then*

$$\|f - \bar{f}\| \leq \frac{\|f_0 - \bar{f}_0\|_0}{1 - 4C(R)\mu(\max\{\|f_0\|_0, \|\bar{f}_0\|_0\})}, \tag{4.2}$$

where $\mu(\tau) = \frac{1-\sqrt{1-8C(R)\tau}}{4C(R)}$ *and* $C(R)$ *is given in Lemma* 4.1.

Proof By the formulas of $f^\sharp$ and $\bar{f}^\sharp$, we have

$$\begin{aligned}(f-\bar{f})^\sharp(t,x,v) = (f_0 - \bar{f}_0)(x,v) + \int_0^t &[Q^\sharp_+(f-\bar{f},f) - Q^\sharp_-(f-\bar{f},f) \\ &+ Q^\sharp_+(\bar{f}, f-\bar{f}) - Q^\sharp_-(\bar{f}, f-\bar{f})]ds.\end{aligned} \tag{4.3}$$

Using Lemma 4.1 and (4.3), it is easy to see that

$$|(f-\bar{f})^\sharp(t,x,v)| \leq |f_0 - \bar{f}_0| + C(R)[2\|f\| + 2\|\bar{f}\|]\|f-\bar{f}\|W(x,v). \tag{4.4}$$

We know

$$f^\sharp(t) \leq u_0^\sharp(t), \qquad \bar{f}^\sharp(t) \leq u_0^\sharp(t),$$

where u_0 is from the beginning condition in the proof of the positivity of the mild solution using Kaniel-Shinbrot iterative scheme [23, 25], i.e.,

$$l_0(t)^\sharp \leq l_1^\sharp(t) \leq u_1^\sharp(t) \leq u_0^\sharp(t), \quad t \in [0,\infty). \tag{4.5}$$

Without loss of generality, we let $u_0^\sharp(t) = \mu(\|f_0\|_0)W(x,v)$, $\bar{u}_0^\sharp(t) = \mu(\|\bar{f}_0\|_0)W(x,v)$, where $\mu(\|f_0\|_0) = \frac{1-\sqrt{1-8C(R)\|f_0\|_0}}{4C(R)}$.

Consequently, $\|f\| \leq \mu(\|f_0\|_0)$, $\|\bar{f}\| \leq \mu(\|\bar{f}_0\|_0)$. Then from (4.4), we get

$$\|f - \bar{f}\| \leq \|f_0 - \bar{f}_0\|_0 + 4C(R)[\max\{\mu(\|f_0\|_0), \mu(\|\bar{f}_0\|_0)\}]\|f - \bar{f}\|. \tag{4.6}$$

From the definition of $\mu(\tau)$, we have

$$\mu(\|f_0\|_0) \leq 2\|f_0\|_0, \quad \mu'(\tau) > 0,$$

which implies $4C(R)[\max\{\mu(\|f_0\|_0), \mu(\|\bar{f}_0\|_0)\}] = 4C(R)\mu(\max\{\|f_0\|_0, \|\bar{f}_0\|_0\}) < 1$. So

$$\|f - \bar{f}\| \leq \frac{1}{1 - 4C(R)\mu(\max\{\|f_0\|_0, \|\bar{f}_0\|_0\})}\|f_0 - \bar{f}_0\|_0. \tag{4.7}$$

This completes the proof. □

A problem The method in this section can be used to the relativistic Boltzmann equation near vacuum [16]. But we can't obtain uniform $L^1(x, v)$-stability of the solution in [16] although the initial data has compact support with respect to x, i.e., the solution $f(t, x, v) \in L^1(x, v)$.

In fact, after taking difference of two solutions and change of variables, the $L^1(x, v)$-stability lays on a Gronwall's inequality:

$$\|(f - \bar{f})(t, x + \hat{v}t, v)\|_{L^1(x,v)} \leq \|f_0 - \bar{f}_0\|_{L^1(x,v)} + \int_0^t \lambda(s)\|(f - \bar{f})(s, x + \hat{v}s, v)\|_{L^1(x,v)} ds,$$

where $\lambda(s)$ is

$$\sup_{x,v} \int_{\mathbb{S}^2_+ \times \mathbb{R}^3_u} q(u, v, \omega)e^{-u_0}[1 + |(x \times u) + s(\hat{v} \times u)|^2]^{-\frac{1+\delta}{2}} du d\omega.$$

One can't obtain time decay of $\lambda(s)$ although $|x| \leq b$, for some b > 0.
On the other hand, if one changes the weight function with respect to x by a exponential one as $e^{-|x \times v|}$, and based on the initial data having compact support in x, by the same procedure the temporal decay rate with respect to t can be obtained because of $e^{-|a+b|} \leq e^{|a|}e^{-|b|}$. Concretely, we can get the temporal decay rate from

$$\sup_{x,v} \int_{\mathbb{S}^2_+ \times \mathbb{R}^3_u} q(u, v, \omega)e^{-(k+1)u_0} \exp(-|(x \times u) + t(\hat{v} \times u)|) du d\omega,$$

so we have the uniform $L^1(x, v)$-stability.

Unfortunately, for this kind of weight function, we can't get global existence of the mild solution as that in [16].

Thus, choosing some new weight functions to obtain both global existence and L^1-stability for the relativistic Boltzmann equation near vacuum is a problem.

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